扩展 KP 类方程有理解与符号计算

Rational solutions to an extended KP-like equation with symbolic computation[1]

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1 引子

非线性微分方程的有理解成为近年来的一个研究热点. 其中怪波解，作为一类特殊有理解，在数学和物理学界引起广泛关注. 这类有理解能够用于刻画海洋学中一些典型的非线性波.

关于积分方程有理解的研究已经形成一套系统的理论.而对于非积分方程有理解的研究尚处于发展过程中 [3]. 在该篇论文中，作者通过广义双线性微分算子和升维得到非积分方程，扩展 KP 类方程. 随后，通过一种多项式生成函数，使用符号计算软件 Maple 得到了该方程的 18 类有理解. 最后，作者给出特殊情况下两类有理解的图像. 该文旨在为多维高阶怪波解的研究做出贡献.

积分方程包括 KdV、Boussinesq、Toda 方程等

2 扩展 KP 类方程的导出

KP 方程

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0, \tag{2.1}$$

在变换 $u = 2[\ln f(x,y,t)]_{xx}$ 下，有双线性导数方程

$$(D_xD_t + D_x^4 + D_y^2)f \cdot f = 0.$$

基于 [2] 介绍的广义微分算子理论，推广上式

许多其他双线性微分方程无法表示为 Hirota 双线性形式

这里引入广义双线性算子 $D_{p,x}, p = 3$

$$\begin{aligned}&(D_{3,x}D_{3,t} + D_{3,x}^4 + D_{3,y}^2 + D_{3,z}^2)f \cdot f \\ &= 2f_{xt}f - 2f_xf_t + 6f_{xx}^2 + 2f_{yy}f - 2f_y^2 + 2f_{zz}f - 2f_z^2 = 0.\end{aligned} \tag{2.2}$$

计算过程:

$$\begin{aligned}\left(D_{p,x}^{n} f \cdot g\right)(x) &= \left(\partial_{x}+\alpha \partial_{x'}\right)^{n} f(x) g\left(x'\right)\Big|_{x'=x} \\ &= \sum_{i=0}^{n} C_{n}^{i} \alpha^{i}\left(\partial_{x}^{n-i} f\right)(x)\left(\partial_{x}^{i} g\right)(x), \\ \left(D_{p,x_{1}}^{n_{1}} D_{p,x_{2}}^{n_{2}} f \cdot g\right)\left(x_{1}, \cdots, x_{l}\right) &= \left(\partial_{x_{1}}+\alpha \partial_{x_{1}'}\right)^{n_{1}}\left(\partial_{x_{2}}+\alpha \partial_{x_{2}'}\right)^{n_{2}} f\left(x_{1}, x_{2}\right) g\left(x_{1}', x_{2}'\right)\Big|_{x_{i}'=x_{i}},\end{aligned}$$

其中 $n, n_i \geq 1, \alpha^i = (-1)^{r(i)}, i = r(i) \bmod p\,(0 \leq r(i) < p, i \geq 0)$

$$\begin{aligned}&D_{3,x} D_{3,t} f \cdot g = f g_{xt} - f_t g_x - f_x g_t + f_{xt} g \\ &D_{3,x}^{2} f \cdot g = f g_{xx} - 2 f_x g_x + f_{xx} g \\ &D_{3,x}^{4} f \cdot g = -f g_{xxxx} + 4 f_x g_{xxx} + 6 f_{xx} g_{xx} - 4 f_{xxx} g_x + f_{xxxx} g\end{aligned}$$

$$\begin{aligned}&\left(D_{3,x} D_{3,t} + D_{3,x}^{4} + D_{3,y}^{2} + D_{3,z}^{2}\right) f \cdot f \\ &= 2 f_{xt} f - 2 f_x f_t + 6 f_{xx}^{2} + 2 f_{yy} f - 2 f_y^{2} + 2 f_{zz} f - 2 f_z^{2}\end{aligned}$$

由贝尔多项式理论，考虑变换

$$u = 2[\ln f(x,y,z,t)]_x = 2\frac{f_x(x,y,z,t)}{f(x,y,z,t)}, \tag{2.3}$$

看到

$$\left[\frac{\left(D_{3,x} D_{3,t}+D_{3,x}^{4}+D_{3,y}^{2}+D_{3,z}^{2}\right) f \cdot f}{f^{2}}\right]_{x} = \left(u_t + \tfrac{3}{2} u_x^2 + \tfrac{3}{8} u^4 + \tfrac{3}{2} u^2 u_x\right)_x + u_{yy} + u_{zz}. \tag{2.4}$$

由 (2.4) 式，(2.2) 式为 $(3+1)$ 维 eKP-like 方程

$$\left(u_t + \tfrac{3}{2} u_x^2 + \tfrac{3}{8} u^4 + \tfrac{3}{2} u^2 u_x\right)_x + u_{yy} + u_{zz} = 0 \tag{2.5}$$

的广义双线性导数方程.

1. 扩展 KP 类方程 (2.5) 式相比标准 KP 方程 (2.1) 式项数更多，非线性程度更高.
2. 如果 f 是 (2.2) 式的解，$u = 2\left[\ln f(x,y,z,t)\right]_x$ 确定 (2.5) 式的一个解.
3. 变换 (2.3) 式能够生成有理解.

3 有理解的符号计算

基于广义双线性导数方程的多项式解，讨论扩展 KP 类方程 (2.5) 式的有理解. 通过符号计算，

$$f = \sum_{i=0}^{4} \sum_{j=0}^{3} \sum_{k=0}^{3} \sum_{l=0}^{5} c_{i,j,k,l} x^i y^j z^k t^l \tag{3.1}$$

为 (2.2) 式的多项式解.

通过变换 (2.3) 式，该多项式解诱导出扩展 KP 类方程的 18 类有理解.

第 1 类

$$u_1 = \frac{2c_{1,1,0,2}}{xc_{1,1,0,2} + c_{0,1,0,2}} \tag{3.2}$$

第 2 类

$$u_2 = \frac{2c_{1,0,0,0}^2}{xc_{1,0,0,0}^2 - tc_{0,1,0,0}^2 + \varepsilon yc_{1,0,0,0}c_{0,1,0,0} + \varepsilon c_{1,0,0,0}c_{0,0,0,0}}, \varepsilon = \pm 1 \tag{3.3}$$

第 3 类

$$u_3 = \frac{2c_{0,1,1,1}^2}{xc_{0,1,1,1}^2 - tc_{0,1,0,2}^2 + \varepsilon zc_{0,1,1,1}c_{0,1,0,2} + \varepsilon c_{0,1,0,1}c_{0,1,0,2}}, \varepsilon = \pm 1 \tag{3.4}$$

第 4 类

$$u_4 = \frac{2c_{1,0,0,0}^2}{p} \tag{3.5}$$

其中

$$\begin{aligned} p = & xc_{1,0,0,0}^2 - t\left(c_{0,0,1,0}^2 + c_{0,1,0,0}^2\right) + yc_{0,1,0,0}c_{1,0,0,0} + zc_{0,0,1,0}c_{1,0,0,0} \\ & + c_{0,0,0,0}c_{1,0,0,0} \end{aligned}$$

第 5 类

$$u_5 = \frac{2p}{q} \tag{3.6}$$

其中

$$\begin{aligned} p =& tc_{0,0,0,2}^3c_{2,0,0,1}c_{0,1,0,1} - 2xc_{0,1,0,1}^3c_{0,0,0,2}c_{2,0,0,1} + yc_{0,0,0,2}^2c_{0,1,0,1}^2c_{2,0,0,1} \\ & + c_{2,0,0,2}^2c_{0,0,0,1}c_{2,0,0,1}c_{0,1,0,1} - c_{0,0,0,2}^3c_{0,1,0,0}c_{2,0,0,1} + c_{0,1,0,1}^5 \\ & - 12c_{0,1,0,1}^3c_{2,0,0,1}^2, \\ q =& txc_{0,0,0,2}^3c_{2,0,0,1}c_{0,1,0,1} - x^2c_{0,1,0,1}^3c_{0,0,0,2}c_{2,0,0,1} + xyc_{0,0,0,2}^2c_{0,1,0,1}^2c_{2,0,0,1} \\ & + x\left(c_{0,0,0,2}^2c_{2,0,0,1}c_{2,0,0,1}c_{0,1,0,1} - c_{0,0,0,2}^3c_{0,1,0,0}c_{2,0,0,1} + c_{0,1,0,1}^5\right. \\ & \left.-12c_{0,1,0,1}^2c_{2,0,0,1}^2\right) - tc_{0,1,0,1}^3c_{0,0,0,2}^2 - yc_{0,1,0,1}^4c_{0,0,0,2} - c_{0,1,0,1}^3c_{0,0,0,1} \\ & + c_{0,0,0,2}^2c_{0,1,0,0}^2c_{0,1,0,0}. \end{aligned}$$

第 6 类

$$u_5 = \frac{2p}{q} \tag{3.7}$$

其中

$$
\begin{aligned}
p =& c_{1,0,0,3}^{2}\left(tc_{1,0,0,3}^{2}c_{1,1,0,2} - 2xc_{1,1,0,2}^{3} + yc_{1,1,0,2}^{2}c_{1,0,0,3} + c_{1,0,0,2}c_{1,0,0,3}c_{1,1,0,2}\right.\\
&\left.- c_{1,0,0,3}^{2}c_{1,1,0,1}\right),\\
q =& txc_{1,0,0,3}^{4}c_{1,1,0,2} - x^{2}c_{1,1,0,2}^{3}c_{1,0,0,3}^{2} + xyc_{1,0,0,3}^{3}c_{1,1,0,2}^{2} + xc_{1,0,0,3}^{2}\left(c_{1,0,0,2}c_{1,0,0,3}\right.\\
&\left.c_{1,1,0,2} - c_{1,0,0,3}^{2}c_{1,1,0,1}\right) + tc_{1,0,0,3}^{3}c_{0,0,0,3}c_{1,1,0,2} + yc_{1,0,0,3}^{2}c_{1,1,0,2}^{2}c_{0,0,0,3}\\
&+ c_{1,1,0,2}^{3}c_{0,0,0,3}^{2} + c_{1,0,0,3}^{2}c_{0,0,0,3}c_{1,0,0,2}c_{1,1,0,2} - c_{1,0,0,3}^{3}c_{0,0,0,3}c_{1,1,0,1} - 12c_{1,1,0,2}^{5}.
\end{aligned}
$$

注意到上述 18 类有理解中，第一类解(3.2)，第二类解(3.3)，第五类解(3.6)，第六类解(3.7)与变量 z 无关.

最后，作者给出第六类解(3.7)，第七类解的特殊情形 $c_{i,j,k,l} = 1+i^2+j^2+k^2+l^2$，此时

$$
u_6 = \frac{204974t - 166012x + 130438y}{195657t - 41503x^2 + 65219xy - 2662x + 59290y - 169804},
$$

$$
u_7 = \frac{12936x^2 - 22}{77616t + 2156x^3 - 11x + 924z + 41496}.
$$

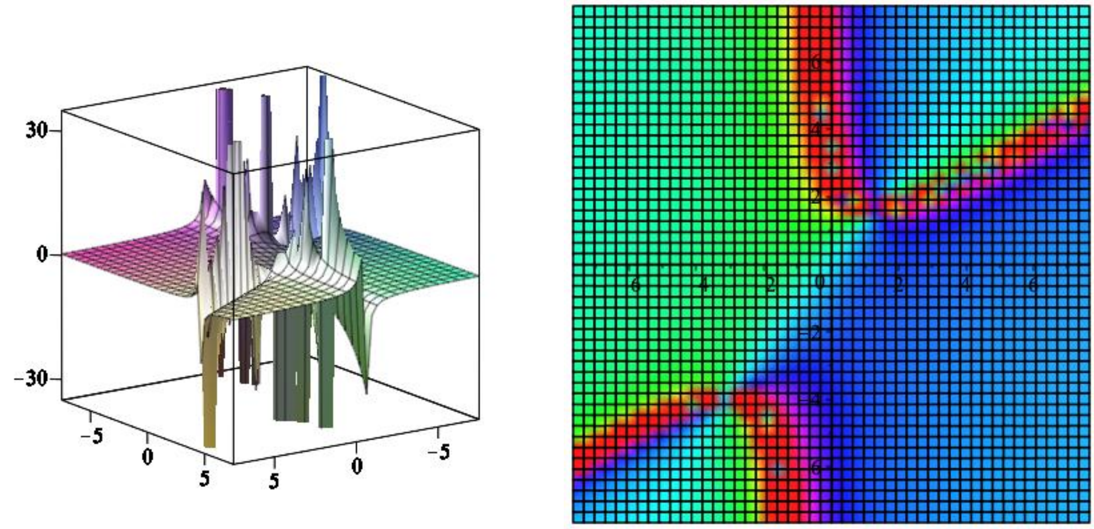

Figure 1: 第六类

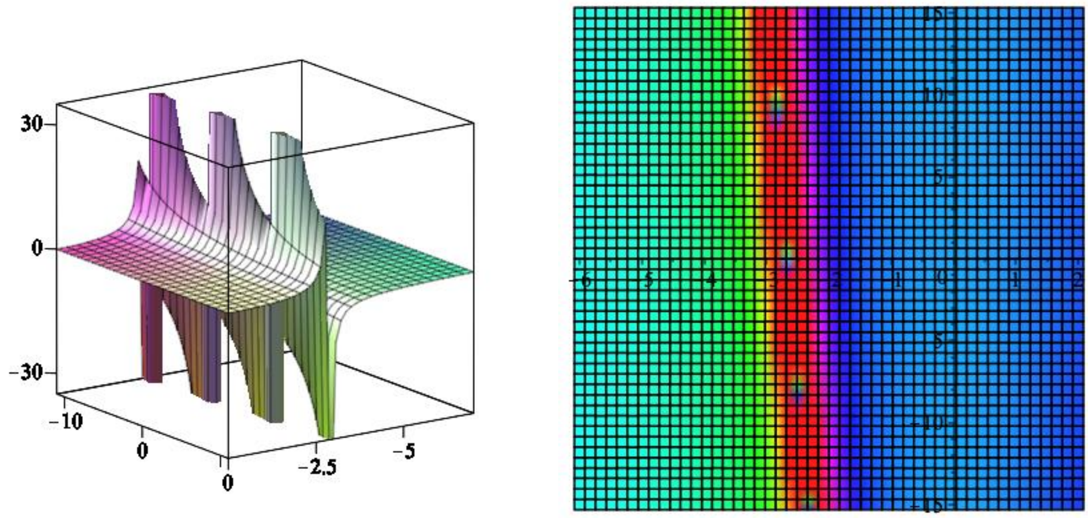

Figure 2: 第七类

4 作业

推导扩展 KP 类方程，即将变换 (2.3) 式代入 (2.2) 式，得到 (2.5) 式.

参考文献

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[2] Wen-Xiu Ma. Bilinear equations, bell polynomials and linear superposition principle. In *Journal of Physics: Conference Series*, volume 411, page 012021. IOP Publishing, 2013.

[3] Yufeng Zhang and Wen-Xiu Ma. A study on rational solutions to a kp-like equation. *Zeitschrift für Naturforschung A*, 70(4):263–268, 2015.